International Journal of Theoretical Physics, Vol. 6, No. 3 (1972), pp. 225-247

A Non-Linear Electromagnetic Theory

FREDERIC BISSHOPP

Division of Applied Mathematics, Brown University, Providence, Rhode Island, 02912

Received: 9 August 1971

Abstract

By adding one term to the Lagrangian density that governs classical electromagnetic theory *via* Maxwell's equations, a parameter-free non-linear generalization can be formulated. Some properties of plane waves, wavepackets and self-focussed beams in the resulting non-linear vector field theory are discussed here.

1. *Introduction*

P. A. M. Dirac (1951, 1952, 1954) has proposed a generalization of classical electromagnetic theory within which he can describe the motion of charged elements of space and the fields due to such charged elements in a unified manner. That is to say, the Lorentz force appears naturally in his theory—it is there from the beginning, rather than as an added hypothesis that extends the universe of Maxwell's equations from the description of electromagnetic wave propagation into the realm of classical electrodynamics. Of course, he has added a different hypothesis, the subsidiary condition [Dirac, 1951, equation (9)],

$$A^i A_i = k^2$$

but it appears very early in the formulation of the theory—at a point where only the vector potential A_i has been defined, where nothing has been said about the existence or non-existence of electrons, and where very little has been said about charge. Even from a strictly classical point of view the introduction of a hypothesis that brings charge in naturally and ultimately subsumes the Lorentz force, if there are charged particles, is attractive, for it erases the particle-field dichotomy and makes it easier to re-examine the notion of constructing a generalized classical electrodynamics that describes the structure of electrons as well as their motions. Dirac's theory is not such a theory, however, in it he has proposed (Dirac, 1954):

> '...that the theory of electrons should be built up from a classical theory of the motion of a continuous stream of electricity rather than the motion of point charges. One then looks upon the existence of discrete electrons as a quantum phenomenon.' (from III)

With no presumption to belittle Dirac's contributions to our understanding of the behavior of electrons, I shall, however, take exception with his proposal. In my opinion, he has suggested that we resort to *quantization* too soon. This article will be devoted to the elucidation of some of the qualitative features of the solutions of a non-linear generalization of Maxwell's equations. The effort here is similar in spirit and intent to Dirac's —his theory is a non-linear generalization of Maxwell's equations (compare Nambu, 1968)—but this is a different theory. The solutions to be discussed here are *classical free particle states*; they are wavepackets that are localized in space and carry with them in their uniform motion a property that can be provisionally identified with internal angular momentum. A related class of solutions describes a possible final state of self-focussed laser beams.

2. $((AA)^2)_4$

The non-linear vector field theory to be discussed here is governed by a variational principle of the form

$$\delta \int (T - V)\, dx^0\, dx^1\, dx^2\, dx^3 = 0 \tag{2.1}$$

with

$$T = -\tfrac{1}{4} g^{ij} k^{kl} (A_{k,i} - A_{i,k})(A_{l,j} - A_{j,l}) \tag{2.2}$$

and

$$V = \pm \tfrac{1}{4} (g^{ij} A_i A_j)^2$$

where repeated covariant–contravariant indices are summed from 0 to 3, indices following a comma denote partial derivatives, and $g^{ij} = g_{ij} = 1, -1,$ or 0 when $i = j = 0$, $i = j \neq 0$ or $i \neq j$. $((AA)^2)_4$ is a natural vector-generalization of the scalar field theory, $(\phi^4)_4$, that is obtained when $T = \tfrac{1}{2} g^{ij} \phi_{,i} \phi_{,j}$ and $V = \pm \tfrac{1}{4} \phi^4$—both are Lorentz-invariant and derived from the same kind of variational principle, and they have the same dimensional group (1, 1) but in terms of structural stability† they are profoundly different. By way of comparison, it may be noted that Dirac's theory is of the form (2.1) with $V = -\tfrac{1}{2} \lambda (g^{ij} A_i A_j - k^2)$, where the Lagrange multiplier that has been introduced to account for the constraint is a scalar field $\lambda(x^k)$ and k^2 is a positive constant.

The field equations of $((AA)^2)_4$ are

$$F^{mn}_{,m} = j^n \tag{2.3}$$

† This is discussed by Bisshopp (1971a).

where

$$F_{ij} \equiv A_{j,i} - A_{i,j} = -F_{ji},$$
$$j^k \equiv \pm(A_l^l A_l)\, A^k$$

and the duality of covariant and contravariant indices is defined, as usual, in terms of the metric $g^{ij} = g_{ij}$.† The four-vector j^k will be provisionally identified with charge-current density, and charge conservation,

$$j^k_{,k} = F^{kl}_{,kl} = 0 \tag{2.4}$$

follows directly from the antisymmetry of F_{ij}, provided the mixed third partial derivatives of A_k are continuous—no situations where this is not the case will be considered here. It follows directly that Maxwell's equations are obtained, if A^k is identified with the vector potential of electromagnetic theory, in the limit where

$$A^k A_k \to 0 \tag{2.5}$$

and it may be noted that they are not quite a weak-field limit, since neither the components of A^k nor their gradients need be small provided the weaker condition (2.5) is satisfied.

Some of the properties of $((AA)^2)_4$ may seem somewhat less obscure when it is recast in the older and more familiar notation where $(A^i) \to (\phi, \mathbf{A})$, $(x^i) \to (t, \mathbf{x})$, and

$$\delta \int \{\tfrac{1}{2}|\nabla\phi + \mathbf{A}_t|^2 - \tfrac{1}{2}|\nabla \times \mathbf{A}|^2 \mp \tfrac{1}{4}(\phi^2 - |\mathbf{A}|^2)^2\}\, d^3x\, dt = 0 \tag{2.6}$$

in any convenient inertial frame. The field equations then are

$$\nabla . \mathbf{E} = \rho \qquad \text{and} \qquad \nabla \times \mathbf{H} - \mathbf{E}_t = \mathbf{j} \tag{2.7}$$

where

$$\mathbf{E} \equiv -\nabla\phi - \mathbf{A}_t, \qquad \mathbf{H} \equiv \nabla \times \mathbf{A},$$
$$\rho \equiv \pm\, \phi(\phi^2 - |\mathbf{A}|^2), \qquad \mathbf{j} \equiv \pm\, \mathbf{A}(\phi^2 - |\mathbf{A}|^2) \tag{2.8}$$

and further relations

$$\nabla . \mathbf{H} = 0, \qquad \nabla \times \mathbf{E} + \mathbf{H}_t = 0, \qquad \rho_t + \nabla . \mathbf{j} = 0 \tag{2.9}$$

are identities. The theory is not gauge invariant, but that does not mean that gauge transformations where $(\phi, \mathbf{A}) \to (\phi + \chi_t, \mathbf{A} - \nabla\chi)$ cannot be used to clarify certain kinds of problems. In the special case where $(\phi, \mathbf{A}) \to (\chi_t, \mathbf{A} - \nabla\chi)$, for example, (2.6) becomes

$$\delta \int \{\tfrac{1}{2}|\mathbf{A}_t|^2 - \tfrac{1}{2}|\nabla \times \mathbf{A}|^2 \mp \tfrac{1}{4}(\chi_t^2 - |\mathbf{A} - \nabla\chi|^2)^2\}\, d^3x\, dt = 0 \tag{2.10}$$

and the charge conservation law,

$$(\chi_t(\chi_t^2 - |\mathbf{A} - \nabla\chi|^2))_t + \nabla . ((\mathbf{A} - \nabla\chi)(\chi_t^2 - |\mathbf{A} - \nabla\chi|^2)) = 0 \tag{2.11}$$

† $F^{mn} = g^{mi} g^{nj} F_{ij}$, $A^k = g^{kl} A_l$, and so on.

is a field equation, rather than an identity that follows from the field equations.

3. *Plane Waves*

Although it is generally impossible to classify and describe the qualitative features of all the solutions in a field theory, we can find at least some of the qualitative features of a field theory by considering certain classes of solutions—similarity solutions—for which the field equations can be reduced to ordinary differential equations. The simplest class of such solutions in $((AA)^2)_4$ is that of the plane waves where

$$A_i(x^k) = a_i(\theta(x^k)), \qquad \theta_{,i} \equiv \omega_i \tag{3.1}$$

and $(\omega_i) \equiv (\omega_0, -\mathbf{k})$ is a *constant* four-vector. The ansatz (3.1) differs from that which is employed at the outset of a Fourier analysis of a linear problem in so far as the functional dependence of $a_i(\omega_0 t - \mathbf{k}.\mathbf{x})$ is not specified in advance to be sinusoidal. Instead, it is to be determined by the field equations, which for solutions of the form (3.1) follow from

$$\begin{aligned} T &= -\tfrac{1}{2}(g^{ij}\omega_i\omega_j g^{kl} a_k' a_l' - (g^{ij}\omega_i a_j')^2) \\ V &= \pm\tfrac{1}{4}(g^{ij} a_i a_j)^2 \end{aligned} \tag{3.2}$$

where the prime denotes differentiation with respect to the argument θ. The field equations for a_i are the Euler equations of

$$\delta \int (T - V)\, d\theta = 0 \tag{3.3}$$

where T and V are expressed as in (3.2)†.

Since $((AA)^2)_4$ is Lorentz-invariant, there is a certain amount of arbitrariness in the choice of the inertial frame in which plane waves are to be described, and we are free to use that choice of frame to simplify the description. Given a four-vector (ω_i), the quantity $\omega^k\omega_k = g^{ij}\omega_i\omega_j$ is invariant under the transformations of the Lorentz group—the relevant feature of it is whether it is positive or negative. In this article we shall consider only the case where $\omega^k\omega_k$ is greater than zero, i.e. the case where the phase velocity of plane waves, $|\omega_0|/|\mathbf{k}|$, is greater than 1. Then there is an inertial frame in which $\mathbf{k} = 0$, i.e. $(\omega_i) = (\omega, 0, 0, 0)$. In that particular frame, which we shall call the *wave-frame*, we shall let

$$(a_i) = (a_0, -\boldsymbol{\alpha}) \tag{3.4}$$

and then

$$\begin{aligned} T &= \tfrac{1}{2}\omega^2|\boldsymbol{\alpha}'|^2 \\ V &= \pm\tfrac{1}{4}(a_0^2 - |\boldsymbol{\alpha}|^2)^2 \end{aligned} \tag{3.5}$$

† Compare Bisshopp (1971b) where this is shown in a more general context, and also Section 4 of the present article.

The Euler equation,

$$(T_{a'_0})' + V_{a_0} = V_{a_0} = \pm a_0(a_0^2 - |\boldsymbol{\alpha}|^2) = 0 \tag{3.6}$$

implies either that $a_0^2 - |\boldsymbol{\alpha}|^2 = g^{ij} a_i a_j = 0$ or that $a_0 = 0$. Since the first option leads directly to a linear theory governed by Maxwell's equations, we naturally choose the second, which has non-linear plane waves governed by

$$\delta \int \{\tfrac{1}{2}\omega^2 |\boldsymbol{\alpha}'|^2 \mp \tfrac{1}{4}|\boldsymbol{\alpha}|^4\}\, d\theta = 0 \tag{3.7}$$

in the wave-frame.

There is an easily distinguished class of solutions of (3.7) that corresponds to the plane-polarized plane waves of electromagnetic theory. The plane-polarized waves where

$$\boldsymbol{\alpha}(\theta) = \hat{n}\phi(\theta) \tag{3.8}$$

and $\hat{n}$ is a unit vector that does not vary with θ are governed by

$$\delta \int \{\tfrac{1}{2}\omega^2 \phi'^2 \mp \tfrac{1}{4}\phi^4\}\, d\theta = 0 \tag{3.9}$$

Now, (3.9) is the equation that governs the plane waves of the $(\phi^4)_4$-theory in the wave-frame, and since those waves have been discussed in some detail by Bisshopp (1971b, 1971c), their properties will not be repeated here. In this article we shall be primarily concerned with rotating waves where

$$(\boldsymbol{\alpha}) = A(\theta)(\cos \Phi(\theta), \sin \Phi(\theta), 0) \tag{3.10}$$

in a coordinate system where the alignment of the positive z-axis is specified by an arbitrary unit vector $\hat{n}$ that does not vary with θ. Since (3.7) implies conservation of angular momentum (in the mechanical analog where $\boldsymbol{\alpha} \leftrightarrow \mathbf{x}$ and $\theta \leftrightarrow t$), (3.10) is the most general form for $(\boldsymbol{\alpha})$, and

$$\delta \int \{\tfrac{1}{2}\omega^2 (A'^2 + A^2 \Phi'^2) \mp \tfrac{1}{4}A^4\}\, d\theta = 0 \tag{3.11}$$

in general.

The Euler equation

$$\omega^2 A^2 \Phi' = l \tag{3.12}$$

is the angular momentum integral in the same mechanical analog. When it is substituted in the final Euler equation, the result,

$$\omega^2 A'' = \frac{l^2}{\omega^2 A^3} \mp A^3 \tag{3.13}$$

is seen to have the energy integral

$$\tfrac{1}{2}\omega^2 A'^2 + U(A) = E \tag{3.14}$$

where

$$U(A) = \frac{1}{2}\frac{l^2}{\omega^2 A^2} \pm \frac{1}{4}A^4 \tag{3.15}$$

is the effective potential. If there are to be bounded solutions of (3.14), $U(A)$ must have a relative minimum, and therefore it is necessary that the plus sign be assigned to V. In the other case where V has the minus sign, there are plane waves when $g^{ij}\omega_i\omega_j$ is less than zero—such waves have phase velocities less than 1.

The plane-polarized waves are recovered formally from (3.14) and (3.15) when $l = 0$, but as can be seen in Fig. 1, the limit where $l \to 0$ is singular. Nevertheless, the plane-polarized wave *is* a rotating wave with vanishingly small angular momentum, as we shall see.

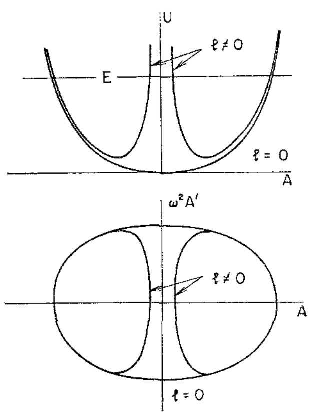


Figure 1—Effective potentials for plane-polarized and rotating plane waves.

The problem of defining the dispersion relations for the rotating waves is somewhat more complicated than it is for the plane-polarized waves [the $(\phi^4)_4$-theory], where the fundamental periodic solution is assigned the period 2π, in accord with the usual convention for specifying the frequency of an oscillation. The resulting dispersion relation is

$$\left(\frac{3\gamma}{4}\right)^2 \omega^2 = E^{1/2} \qquad \left[\gamma = \frac{(\frac{1}{2})!(-\frac{3}{4})!}{\pi(\frac{3}{4})!} \approx 1{\cdot}1128\right] \tag{3.16}$$

in the wave-frame.† Given the values of ω^2, l^2 and E, the solution $\boldsymbol{\alpha}(\theta)$ of (3.7) is, in general, a doubly periodic function of θ. The libration period $2\pi r$ is the period of the fundamental periodic solution defined by (3.14) and (3.15), and the rotation period $2\pi s$ can be defined as the corresponding advance of Φ according to (3.12) when θ increases by $2\pi r$. A rather obvious convention we might adopt to define dispersion relations is that $r = 1$, thus defining ω to be the libration frequency. Another possibility is to consider only strictly periodic waves where $r = 1/q$, $s = p/q$, p and q are integers

† Compare Bisshopp (1971b, 1971c).

(relatively prime defines a fundamental oscillation), and the period of $\boldsymbol{\alpha}$ is 2π. Neither of these is a very convenient convention for specifying the dispersion relations, however, and for reasons that will become clearer as the detailed behavior of the rotating waves is unfolded, we shall adopt the convention that $s = \pm r$, i.e. the libration and rotation periods are equal except for a sign that will be determined by the sign of l. The advance of Φ during a cycle of the libration is then equal in magnitude to the advance of θ, and ω is the mean rate of rotation of $\boldsymbol{\alpha}$ in radians per unit time.

The dispersion relations can be expressed as follows. Given ω^2 and l^2, let E be large enough for (3.14) to have real, periodic solutions, and define the *action*

$$J(\omega^2, l^2, E) \equiv \oint \omega^2 A'\, dA = \oint [2\omega^2(E-U)]^{1/2} |dA| \tag{3.17}$$

where the integral is over one period of the libration. Then

$$\oint d\theta = \oint \frac{dA}{A'} = \oint \frac{\omega^2 |dA|}{[2\omega^2(E-U)]^{1/2}} = J_E = 2\pi r \tag{3.18}$$

and

$$\oint d\Phi = \oint \frac{l}{\omega^2 A^2} \frac{dA}{A'} = \oint \frac{l\,|dA|}{A^2[2\omega^2(E-U)]^{1/2}} = -J_l = \pm 2\pi r \tag{3.19}$$

Evidently, the dispersion relations $E = E_r(\omega)$ and $l = l_r(\omega)$ are dependent upon the wave form (i.e. on the value of r), and they are implicit in (3.18) and (3.19). The problem of determining the functions E_r and l_r cannot be expected to be a very pretty one in a general vector field theory that carries rotational waves, but since $((AA)^2)_4$ has a dimensional group that has not been used yet, it can be simplified considerably here. Let

$$A = \pm \left|\frac{l}{\omega}\right|^{1/3} \alpha \qquad \text{and} \qquad E = \left|\frac{l}{\omega}\right|^{4/3} e \tag{3.20}$$

Then

$$J = 2|l| \int_{\alpha_m}^{\alpha_M} \left[2\left(e - \frac{1}{2\alpha^2} - \frac{1}{4}\alpha^4\right)\right]^{1/2} d\alpha \tag{3.21}$$

where the limits of integration are the lesser and greater positive zeroes of the integrand, and the implicit dispersion relations are

$$J_E = \left|\frac{l}{\omega^4}\right|^{-1/3} j'(e) = 2\pi r \tag{3.22}$$

and

$$J_l = \frac{|l|}{l}\left(j(e) - \tfrac{4}{3} e j'(e)\right) = \mp 2\pi r \tag{3.23}$$

where the prime denotes an ordinary derivative of

$$j(e) \equiv 2 \int_{\alpha_m}^{\alpha_M} \left[2\left(e - \frac{1}{2\alpha^2} - \frac{1}{4}\alpha^4\right)\right]^{1/2} d\alpha$$

$$= 4e^{3/4} \int_{\alpha_m}^{\alpha_M} \left(1 - a^4 - \frac{1}{4a^2 e^{3/2}}\right)^{1/2} da \qquad (3.24)$$

The function $j(e)$ is defined on the interval $\frac{3}{4} \leqslant e \leqslant \infty$; the second form of it $[\alpha = (4e)^{1/4} a]$ is particularly useful for the determination of the asymptotic behavior of the dispersion relations in the limit where $e \to \infty$. After the choice of the $\pm$ sign has been made in accord with the sign of l, (3.24) can be solved (numerically) for $e(r)$. Then

$$l = \pm \left(\frac{j'(e(r))}{2\pi r}\right)^3 \omega^4$$

and (3.25)

$$E = e(r) \left(\frac{j'(e(r))}{2\pi r}\right)^4 \omega^4$$

The analysis of (3.25) can be almost entirely completed in terms of the asymptotic results:

$$j(e) = \pi\sqrt{\tfrac{2}{3}}(e - \tfrac{3}{4}) - 0((e - \tfrac{3}{4})^2)$$

and (3.26)

$$\tfrac{4}{3}ej' - j = \pi\sqrt{\tfrac{2}{3}}[1 + \tfrac{1}{3}(e - \tfrac{3}{4}) - 0((e - \tfrac{3}{4})^2)]$$

for $e \to \frac{3}{4}$ from above, and for $e \to \infty$,

$$j(e) = \pi\gamma e^{3/4} - \pi + 0(e^{-3/4})$$

$$\tfrac{4}{3}ej' - j = \pi - 0(e^{-3/4}) \qquad (3.27)$$

where

$$\gamma = \frac{4}{\pi} \int_0^1 (1 - a^4)^{1/2} da$$

$$= \frac{(\frac{1}{2})!(-\frac{3}{4})!}{\pi(\frac{3}{4})!} \approx 1{\cdot}1128 \qquad (3.28)$$

The results (3.26) follow quite easily from a straightforward expansion in powers of $(e - \frac{3}{4})$. To obtain (3.27) is a more complicated problem—Fig. 2 is a schematic diagram that indicates the sources of the leading contributions to the evaluation of j when e is large.

In (3.26) and (3.27) both the order and the sign of the first neglected term are indicated, and it is strongly suggested that $\frac{4}{3}ej' - j$ increases monotonically from $\pi\sqrt{\frac{2}{3}}$ to π, while its slope decreases monotonically from $\pi\sqrt{\frac{2}{3}}/3$

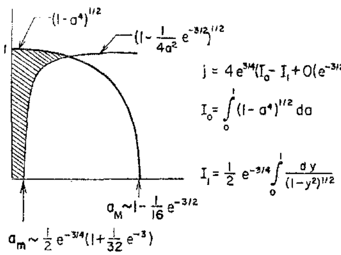


Figure 2—Leading terms of j when $e \to \infty$.

to 0 as e increases from $\frac{3}{4}$ to ∞. Figure 3 is a graph of it that includes the information in (3.26) and (3.27), numerical computations at $e = 1, 2, 5$ and 9 and an estimate of the term of $0(e^{-3/4})$.

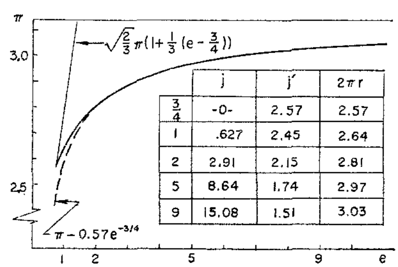
Evidently, there corresponds to each value of $r(e)$ in the range $1/\sqrt{6} \leqslant r \leqslant \frac{1}{2}$ a single value of e and a unique member of the set of nested torii defined in the $\alpha\cos\Phi$, $\alpha\sin\Phi$, α'-space by the relation,

$$\frac{1}{2}\alpha'^2 + \frac{1}{2\alpha^2} + \frac{1}{4}\alpha^4 = e, \qquad \tfrac{3}{4} \leqslant e \leqslant \infty \tag{3.29}$$

The trace of the corresponding solution winds about the torus, and, depending on whether r is rational ($=p/q$) or irrational, either closes upon itself after p rotations and q librations to define a strictly periodic solution, or fails to close, and ultimately passes arbitrarily near every point on the torus. In the limit where r approaches $1/\sqrt{6}$ the torii collapse to the degenerate one, which is the circle defined by $e = \frac{3}{4}$, the libration disappears, and the result is the non-linear analog of the familiar circularly polarized electromagnetic wave. Our convention for specifying the dispersion relations coincides with the usual one in this case, and thus the dispersion relations

$$l = \pm\omega^4 \qquad \text{and} \qquad E = \frac{3\omega^4}{4} \tag{3.30}$$

for circularly polarized waves are obtained by setting $r = 1/\sqrt{6}$ in (3.25). In the other limit where $r \to \frac{1}{2}$ and $e \to \infty$; $j' \to 0$, $l/\omega^4 \to 0$, $E \to (3\gamma/4)^4\omega^4$,



	j	j'	$2\pi r$
$\frac{3}{4}$	-0-	2.57	2.57
1	.627	2.45	2.64
2	2.91	2.15	2.81
5	8.64	1.74	2.97
9	15.08	1.51	3.03

Figure 3—$(\frac{4}{3}ej'(e) - j(e)) \equiv 2\pi r(e)$.

the solution approaches the plane-polarized wave, and the dispersion relation (3.16) is recovered.

It may be of interest to see how the plane-polarized wave, for which $A = 0$ is not in any way a singular feature of the wave form, can be a limiting case of rotating waves, for which $A = 0$ is singular. The wave form of a rotating wave can be represented by plotting the projection of its trace on the $\alpha \cos \Phi$, $\alpha \sin \Phi$-plane. Figure 4 is a schematic diagram of two rotating waves in which the positions and magnitudes of the maxima and minima of α are represented faithfully (five, incidentally, is the lowest value of p

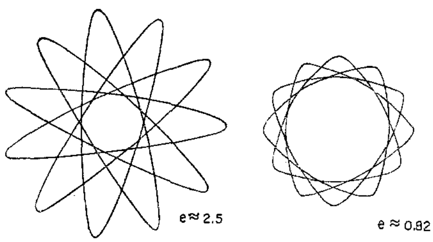


Figure 4—Paradigms of the $\frac{5}{11}$ and $\frac{5}{12}$ rotating waves.

for which there are three strictly periodic waves, including the $\frac{5}{10}$-fourth harmonic of the plane-polarized wave). As $r \to \frac{1}{2}$ and $e \to \infty$, the value of the minima of α approaches zero, and the asymptotic description of the rotating wave as a plane-polarized wave which has a plane of polarization that rotates about the polarization vector $\hat{\pi}$ in it becomes better and better. At $r = \frac{1}{2}$ the rate of rotation of the plane of polarization is zero. Another way to look at it is to examine the behavior of $A(\theta)$ and $\Phi(\theta)$ for a sequence of values of r that approaches $\frac{1}{2}$. As r increases, $A(\theta)$ approaches the function $|A|(\theta)$, which is the magnitude of the amplitude of the plane-polarized wave (it has corners at $|A| = 0$), and $\Phi(\theta)$ approaches a piecewise constant function that increases (or decreases) discontinuously by π at each zero of A. The two views of the limit are not contradictory.†

The paradigms of the rotating waves (Fig. 4 and others) are to be contrasted with the Lissajous figures that are commonly cited as examples of periodic solutions of linear, two-dimensional, vector wave equations. Since superposition of solutions is not allowed in $((AA)^2)_4$, the Lissajous figures are *not* the traces of limiting wave forms in the non-dispersive limit where $\omega \to 0$. This brings to mind the question of whether or not $((AA)^2)_4$ contains non-linear analogs of the elliptically polarized waves of electromagnetic theory. (Any respectable candidate for a non-linear electromagnetic theory had better contain them, since they are regularly observed.) The answer is *yes*, but to see them we must take the plane waves out of the wave-frame, as follows.

† All this, and much more, was well known to the nineteenth-century mathematicians who studied the properties of elliptic functions.

The wave-frame, it may be recalled, was defined as the inertial frame in which the frequency-wavenumber vector is $(\omega, \mathbf{0})$. In the laboratory frame the corresponding quantity is $(\omega_i) = (\omega_0, -\mathbf{k})$, and since the plus sign has been chosen for V [equation (2.2)], the phase velocity of every finite amplitude plane wave is greater than 1 and the velocity of the wave-frame relative to the laboratory frame (the reciprocal of the phase velocity) is less than 1. Given the direction of $\mathbf{k}$, which we shall take to be the positive z-axis of the laboratory frame, and any convenient orientation of the x- and y-axes, the direction of the polarization vector can be specified in terms of Euler angles, Θ and Ψ, and the rotation angle $\Phi(\theta)$ can be measured from the line of nodes, all as in Fig. 5.

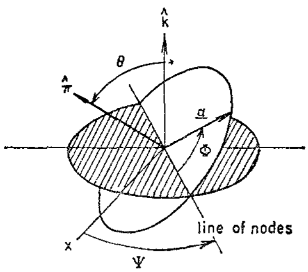


Figure 5—Euler angles.

Since $\hat{\pi}$ is fixed for a plane wave, Θ is constant and Ψ can be set equal to zero (without loss of generality), to obtain the result that

$$(\bar{a}^i) = A(\theta)(0, \cos\Phi(\theta), \cos\Theta\sin\Phi(\theta), \sin\Theta\sin\Phi(\theta)) \tag{3.31}$$

in the frame that moves with the velocity k/ω_0 on the positive z-axis of the laboratory frame. In the laboratory frame

$$(a^i) = (a^0, \boldsymbol{\alpha}) = A\left(\frac{k\sin\Theta\sin\Phi}{\omega_0\sqrt{(1-k^2/\omega_0^2)}}, \cos\Phi, \cos\Theta\sin\Phi, \frac{\sin\Theta\sin\Phi}{\sqrt{(1-k^2/\omega_0^2)}}\right) \tag{3.32}$$

and since

$$A^i(x^k) = a^i(\omega_0 t - kz) \tag{3.33}$$

the fields observed in the laboratory frame are

$$\mathbf{H} = \nabla\times\mathbf{A} = -\mathbf{k}\times\boldsymbol{\alpha}' \quad \text{and} \quad \mathbf{E} = -\nabla A^0 - \mathbf{A}_t = \mathbf{k}a^{0\prime} - \omega_0\boldsymbol{\alpha}' \tag{3.34}$$

Now, $\mathbf{H}$ is clearly transverse to $\mathbf{k}$, and the transverse part of $\mathbf{E}$ is

$$\mathbf{E} - \hat{k}(\hat{k}.\mathbf{E}) = \frac{-\omega_0}{k}\hat{k}\times\mathbf{H} \tag{3.35}$$

in accord with results obtained from Maxwell's equations, but there is also a longitudinal component,

$$\hat{k}.\mathbf{E} = \frac{-\omega_0}{|\omega_0|}\sqrt{(\omega_0^2 - k^2)}[A \sin \Phi]' \sin \Theta \tag{3.36}$$

in the laboratory frame. Given a wave form that is fixed by the specification of Θ, Ψ and e [or $r(e)$], then

$$A = \pm \left(\frac{4E}{\omega^4}\right)^{1/4} |\omega|\, a \tag{3.37}$$

which follows from (3.20), (3.25) and the substitution $\alpha = (4e)^{1/4}a$ that was used to obtain the second form of (3.24), implies an ordering of the magnitudes of the components of the electric field in the nondispersive limit where $\omega_0^2 - k^2 \to 0$. Since a is bounded above by 1 and E/ω^4, above and below by $\frac{3}{4}$ and $(3\gamma/4)^4$ for all rotational waves, the results

$$A = 0[\sqrt{(\omega_0^2 - k^2)}], \quad |\mathbf{E}| = 0[\sqrt{(\omega_0^2 - k^2)}] \quad \text{and} \quad |\hat{k}.\mathbf{E}| = 0(\omega_0^2 - k^2) \tag{3.37}$$

imply that the fields of all the rotating waves are transverse in the laboratory frame in the non-dispersive limit. The case where $e = \frac{3}{4}$ supplies the non-linear analog of the elliptically polarized waves of electromagnetic theory, and in addition there are more complicated rotating, elliptical waves of the kind illustrated in Fig. 6.

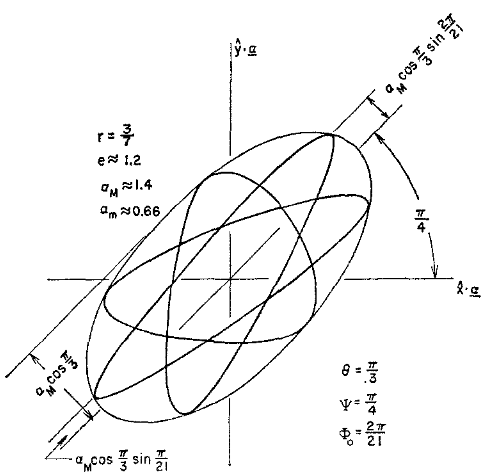


Figure 6—Paradigm of the transverse part of the $\left(\frac{\pi}{3}, \frac{\pi}{4}, \frac{2\pi}{21}, \frac{3}{7}\right)$-wave.

An interesting limiting case of the rotating, elliptical waves is that where Θ is $\pi/2$ and the wave is seen 'edge on', so to speak, in the laboratory frame. In the case where $e = \frac{3}{4}$ we obtain a *different* plane-polarized wave with a sinusoidal wave form in place of the non-linear wave form determined by the $(\phi^4)_4$-theory and with the dispersion relation $E = 3\omega^4/4$ of the circularly polarized wave. Even more interesting, is the 'edge on' wave with $e \neq \frac{3}{4}$ for which the wave form has 'beats', as illustrated in Fig. 7.

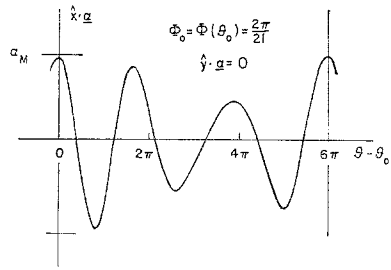


Figure 7—The transverse part of the $\left(\frac{\pi}{2}, 0, \frac{2\pi}{21}, \frac{3}{7}\right)$-wave form.

Thus, we have, altogether, a four-parameter family of nonlinear analogs of electromagnetic waves. In the $(\Theta, \Psi, \Phi_0, r)$-wave, the Euler angles fix the orientation of $\hat{\pi}$ relative to $\hat{k}$ and the orientation of the line of nodes relative to any fixed unit vector $\hat{\imath}$ perpendicular to $\hat{k}$, Φ_0 is the angle between the line of nodes and any ray defined by a recognizable feature of the wave form, and r specifies the doubly periodic structure. The implied twofold degeneracy that follows from the choice of sign that can be made in the specification of l can be formally removed by always assigning the plus sign to l, while Θ and Ψ are both permitted to take on all values. Then, however, the surface traced by the endpoints of polarization vectors that correspond to different solutions is a double covering of the unit sphere, since the full range $(0,2\pi)$ of Ψ is required to specify the orientation of the ellipse within which the transverse part of the rotating wave is inscribed (*not* symmetrically in general), at least the range $(0,\pi)$ of Θ is required to account for the tracing of the paradigms of the transverse components in *either* direction (this much covers the sphere once), and the $(\Theta, \Psi, \Phi_0, r)$- and $(-\Theta, \Psi, \Phi_0, r)$-waves have identical transverse components while their longitudinal (and time-like) components have opposite signs.† The sign of A is immaterial, since $A \to -A$ is equivalent to $\Theta \to -\Theta$ and $\Psi \to \Psi \pm \pi$.

Actually, a little more, of a speculative nature, can be said here. If we confine our attention to those waves for which r is rational ($=p/q$) and measure Φ_0 from the line of nodes to the nearest maximum of α, then when q is even the transverse part of the rotating wave is symmetric after all

† A relation between double covering of the unit sphere and spinors *via* the Cayley-Klein parameters is discussed by Goldstein (1953, Sections 4 and 5).

(when $\Psi \to \Psi \pm \pi$), the range of Ψ is halved to $(0, \pi)$, and the double covering of the unit sphere is lost—when q is odd the asymmetry is there. Thus we may provisionally identify the wavepackets of the p/q rotating waves with prototypical models of *classical* bosons and fermions, depending on whether q is even or odd.

4. *Wavepackets*

In this section we shall isolate and discuss a very special class of modulated waves, saving the discussion of more general modulated waves, which is beyond the scope of this paper, for later. Let us consider the class of solutions in $((AA)^2)_4$ where

$$\begin{aligned}(\phi, \mathbf{A}) = A(\omega t, \mathbf{x})(0, \cos \Phi(\omega t, \mathbf{x}), \sin \Phi(\omega t, \mathbf{x}), 0)&\\ A(\omega t + 2\pi r, \mathbf{x}) = A(\omega t, \mathbf{x})&\\ \Phi(\omega t + 2\pi r, \mathbf{x}) = \Phi(\omega t, \mathbf{x}) + 2\pi r&\end{aligned} \tag{4.1}$$

in some inertial frame, the wave-frame. This includes the plane waves of the previous section in the special case where A and Φ are independent of $\mathbf{x}$, and it describes a modulated wave with a carrier wave form that does not differ qualitatively from that of a $2\pi r$-periodic plane wave. The frequency ω and the wave form parameter r are to be fixed, and it may be noted that the assumption that the wave-frame is inertial implies it is not rotating—hence the Euler angles θ, Ψ and Φ_0 of the carrier are fixed as well. Although (4.1) implies serious restrictions on the class of solutions under consideration, the Euler equations that govern them are still rather complicated, and even the problem of proving the existence of wavepacket-like solutions of them looks rather difficult. What will *in fact* be discussed here is an approximate theory of them, in which the approximate $\mathbf{x}$-dependence of solutions is determined by a kind of coefficient averaging.

Before the approximate theory is presented, it will be helpful to cite two representation theorems that provide a structure within which perturbation theories and other approximate theories can be imbedded. The first theorem has to do with representations of the form

$$A^i(x^k) = \tilde{A}^i(P(x^k), x^k) \tag{4.2}$$

of solutions in field theories governed by variational principles of the form

$$\delta \int \mathscr{L}(A^i, A^i_{,j})\, d^4 x = 0 \tag{4.3}$$

If we add the stipulation that the functions $\tilde{A}^i$ are periodic or almost periodic functions of their first arguments, then (4.2) is a generalized phase-amplitude representation in which the phase $P(x^k)$ is displayed explicitly and the amplitude is contained implicitly in the x^k-dependence. In the discussion of phase-amplitude representations it is convenient to regard the functions $\tilde{A}^i$ as functions of five independent variables, i.e. $\tilde{A}^i(\theta, x^k)$. Then

$$\tilde{A}^i(x^k) = \tilde{A}^i(\theta, x^k)|_{\theta=P(x^k)} \quad \text{and} \quad A^i_{,j}(x^k) = (P_{,j}\tilde{A}^i{}_\theta + \tilde{A}^i_{,j})|_{\theta=P(x^k)} \tag{4.4}$$

where the subscript θ denotes a partial derivative with respect to the first argument. Now let us define

$$\tilde{\mathscr{L}}(\tilde{A}^i, \tilde{A}^i{}_\theta, \tilde{A}^i_{,j}; P_{,j}) \equiv \mathscr{L}(\tilde{A}^i, P_{,j}\tilde{A}^i{}_\theta + \tilde{A}^i_{,j}). \tag{4.5}$$

Then we have

Theorem 1

If $\tilde{A}^i(\theta, x^k)$ is a solution of the Euler equations of

$$\delta \int \tilde{\mathscr{L}}(\tilde{A}^i, \tilde{A}^i{}_\theta, \tilde{A}^i_{,j}; P_{,j})\, d^4x\, d\theta = 0 \quad (\text{given } P_{,j}) \tag{4.6}$$

then $\tilde{A}^i(P(x^k), x^k)$ is a solution of the Euler equations of (4.3).

The theorem can be proved by writing out the Euler equations of (4.3) under the representation (4.2), using the chain rule for partial differentiation as in (4.4). The further relations

$$\tilde{\mathscr{L}}_{\tilde{A}^i} = \mathscr{L}_{\tilde{A}^i}, \qquad \tilde{\mathscr{L}}_{\tilde{A}_\theta{}^i} = P_{,j}\mathscr{L}_{\tilde{A}^i{}_{,j}}, \qquad \tilde{\mathscr{L}}_{\tilde{A}^i{}_{,j}} = \mathscr{L}_{\tilde{A}^i{}_{,j}} \tag{4.7}$$

imply that the result is the set of Euler equations of (4.6), all evaluated at $\theta = P(x^k)$. What the theorem provides is a general framework within which averages over the phase of a modulated wave can be computed without the necessity for computing the phase itself. The functions $P_{,i}$ can be replaced by a set of functions ω_i if we append the integrability conditions

$$\omega_{i,j} = \omega_{j,i} \tag{4.8}$$

The functions $(\omega_i) = (\omega_0, -\mathbf{k})$ are to be interpreted as the local frequency and wave-number vector, as in geometrical optics—$P(x^k)$ need not be computed from them to evaluate averages over the phase, since such averages are simply averages over the fifth independent variable θ.

The variational principles (4.3) and (4.6) are to be interpreted at this stage of the development of $((AA)^2)_4$ and similar field theories merely as a convenient way to write a set of partial differential equations that happen to be Euler equations—nothing has been said yet about boundary conditions or behavior of the fields at large distances from the neighborhood of a localized solution. Such questions about boundary conditions on the fields A^i will be deferred until more is known about properties of some solutions—only the conditions on the θ-dependence of $\tilde{A}^i$ need be discussed at this point. Theorem 1, it may be noted, is formulated entirely in terms of Euler equations, and nothing need be said about boundary conditions to derive it. If, however, (4.6) is to be something more useful than an unnecessary complication of (4.3), it makes sense to restrict the θ-dependence of $\tilde{A}^i$ in a manner that reflects some known properties of the field theory. For $((AA)^2)_4$ the existence of periodic and almost periodic plane waves suggests the restriction of (4.6) to the class of solutions where the θ-dependence of $\tilde{A}^i$ is almost periodic.†

Given a restriction of the fields $\tilde{A}^i$ to the class of almost periodic functions, we can now look into the possibility of formulating direct methods for the solution of (4.6). What is to be discussed is in some sense a generalization of the Rayleigh-Ritz method (a special case of the Galerkin method for problems governed by a variational principle, also called the variational method in some contexts) where the solution is expressed in terms of representations of the form

$$\tilde{A}^i(\theta, x^k) = \bar{A}^i(\theta; Q^\alpha(x^k)) \tag{4.9}$$

In the representation (4.9) the set $\{Q^\alpha(x^k)\}$ is a countable, generally infinite, set of parameters, in terms of which the θ-dependence of $\bar{A}^i$ is *specified* at each point (x^k). For example, there is in general a trigonometric series of the form

$$f(\theta) \sim \sum_1^\infty a_n \exp(i\lambda_n \theta) \tag{4.10}$$

associated with every almost periodic function, but the question of convergence of the series to the function has to be treated rather carefully.‡ For solutions of the form (4.1), the fields $\tilde{A}^i$ belong to a restricted class of almost periodic functions, and the question of convergence is less troublesome. The phase is $\theta = \omega t$, and the parameters Q^α which depend only on $\mathbf{x}$ can be taken to be the Fourier coefficients of

$$\begin{aligned} A(\theta, \mathbf{x}) &= \sum_0^\infty A_n(\mathbf{x}) \cos\left(\frac{n\theta}{r}\right) \\ \Phi(\theta, \mathbf{x}) &= \theta + \sum_1^\infty \Phi_n(\mathbf{x}) \sin\left(\frac{n\theta}{r}\right) \end{aligned} \tag{4.11}$$

Solutions of (4.6) are at least C^2 functions of θ, and the series (4.11) are convergent, with the possible exception of the limiting case where $r = \frac{1}{2}$.

In any case, given a representation of the form (4.9), we can in principle evaluate

$$\begin{aligned} \bar{\mathscr{L}} &\equiv \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathscr{L}(\bar{A}^i, P_{,j}\, \bar{A}^i{}_\theta + \sum Q^\alpha_{,j} \bar{A}^i_{Q^\alpha})\, d\theta \\ &= \bar{\mathscr{L}}(Q^\alpha, Q^\alpha_{,j}, P_{,j}) \end{aligned} \tag{4.12}$$

† Of course, $\mathscr{L}$ really stands for the Lagrangian density of $((AA)^2)_4$ in this section. If similar remarks are to be applied to other field theories, it makes good sense to look first to see if they have almost periodic plane waves.

‡ Compare Besicovitch (1954).

The second representation theorem is

Theorem 2

Given a representation of the form (4.9) of an almost periodic solution of (4.6); the parameters Q^α and the phase P of it are related by the Euler equations of

$$\delta \int \bar{\mathscr{L}}(Q^\alpha, Q^\alpha_{,j}, P_{,j})\, d^4 x = 0 \tag{4.13}$$

The proof of Theorem 2 follows by the use of (4.12) to perform a direct evaluation of

$$(\bar{\mathscr{L}}_{Q^\alpha_{,j}})_{,j} - \bar{\mathscr{L}}_{Q^\alpha} = \lim_{T\to\infty} \frac{1}{2T}\left\{ \int_{-T}^{T} \bar{A}^i_{Q^\alpha}\{(\mathscr{L}_{\bar{A}^i_\theta})_\theta + (\mathscr{L}_{\bar{A}^i_{,j}})_{,j} - \mathscr{L}_{\bar{A}^i}\}\, d\theta - (\bar{A}^i_{Q^\alpha} \mathscr{L}_{\bar{A}^i_\theta})|^{T}_{-T} \right\} = 0 \tag{4.14}$$

and the observation that

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \mathscr{L}_\theta\, d\theta = \lim_{T\to\infty} \frac{1}{2T} (\mathscr{L})\Big|_{-T}^{T} = (\bar{\mathscr{L}}_{P_{,j}})_{,j} + \lim_{T\to\infty} \frac{1}{2T}\left\{ \int_{-T}^{T} \bar{A}^i{}_\theta [\mathscr{L}_{\bar{A}^i} - (\mathscr{L}_{\bar{A}^i_\theta})_\theta - (\mathscr{L}_{\bar{A}^i_{,j}})_{,j}]\, d\theta + (\bar{A}_\theta{}^i \mathscr{L}_{\bar{A}^i_\theta})\Big|_{-T}^{T} \right\} = (\bar{\mathscr{L}}_{P_{,j}})_{,j} = 0 \tag{4.15}$$

Equation (4.15) indicates that the phase $(P(x^k))$ cannot be chosen arbitrarily in a solution of the form (4.9)—instead, it is to be determined concurrently with the parameters $Q^\alpha(x^k)$. In effect (4.15) is an integrability condition, necessary if the fields $\tilde{A}^i$ are to be bounded, almost periodic, functions of θ. Here again, $P_{,i}$ can be replaced by $\omega_i(x^k)$ if we append the integrability conditions (4.8).

Theorem 2 provides the structure within which the truncated approximations of the Rayleigh-Ritz method can be formulated. The sequence of approximations is defined by including one more of the parameters Q^α in each successive member of it. That the approximations are improved as the number of parameters is increased, will be taken as an article of faith here, but it may be noted that some recent work on the Galerkin method offers the promise that convergence and existence of solutions can be established in this framework.†

Here we shall discuss some properties of the solutions of the roughest (one term) approximation that is exact in at least one case (the plane wave). Equation (4.1) is to be replaced by

$$(\phi, \mathbf{A}) \sim A(\mathbf{x})\, a_r(\omega t)(0, \cos \Phi_r(\omega t), \sin \Phi_r(\omega t), 0) \tag{4.16}$$

† Compare Cesari (1964).

where $a_r(\theta)$ and $\Phi_r(\theta)$ are the amplitude and angle of the $2\pi r$-periodic plane wave that was discussed in Section 3. If $a_r(\theta)$ is defined in accord with (3.37), then $A=(4E)^{1/4}$ for the plane wave. For approximations of the form (4.16)

$$\begin{aligned}\bar{\mathscr{L}} \sim \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} &\{\tfrac{1}{2}\omega^2 A^2(a_r'^2+a_r^2\Phi_r'^2)-\tfrac{1}{4}a_r^4A^4-\tfrac{1}{2}a_r^2[A_z^2\\ &+\tfrac{1}{2}(A_x^2+A_y^2)-A_xA_y\sin 2\Phi_r+\tfrac{1}{2}(A_y^2-A_x^2)\cos 2\Phi_r]\}\,d\theta\\ =\tfrac{1}{2}\omega^2\alpha_r A^2&-\tfrac{1}{2}\beta_r(A_z^2+\tfrac{1}{2}(A_x^2+A_y^2))-\tfrac{1}{4}\delta_r(A_y^2-A_x^2)\\ &-\tfrac{1}{4}\gamma_r A^4\end{aligned}\qquad(4.17)$$

where

$$\alpha_r=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}(a_r'^2+a_r^2\Phi_r'^2)\,d\theta=\frac{1}{2\pi r}\int_{-\pi r}^{\pi r}(a_r'^2+a_r^2\Phi_r'^2)\,d\theta$$

$$\beta_r=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}a_r^2\,d\theta=\frac{1}{2\pi r}\int_{-\pi r}^{\pi r}a_r^2\,d\theta \qquad (4.18)$$

and

$$\gamma_r=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}a_r^4\,d\theta=\frac{1}{2\pi r}\int_{-\pi r}^{\pi r}a_r^4\,d\theta$$

are positive numbers that can easily be evaluated, given r. The vanishing of the integral of $a_r^2\sin 2\Phi_r$ can easily be arranged by assigning the Euler angle Φ_0 in such manner that $\theta=0$ is a relative maximum of a_r—then Φ_r and $\sin 2\Phi_r$ are odd functions of θ, a_r and a_r^2 are even functions, and the integral vanishes. The integral

$$\delta_r=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}a_r^2\cos 2\Phi_r\,d\theta \qquad (4.19)$$

varies with r in a manner that seems rather insane at first sight, but really is not. To evaluate it we may first divide the interval $(-\infty,\infty)$ into libration periods as $(\cdots -3\pi r,-\pi r,\pi r,3\pi r\cdots)$, and then write

$$\begin{aligned}\delta_r&=\lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\frac{1}{2\pi r}\int_{-\pi r}^{\pi r}a_r^2\cos 2(\Phi_r+2n\pi r)\,d\theta\\ &=\frac{1}{2\pi r}\int_{-\pi r}^{\pi r}a_r^2\cos 2\Phi_r\,d\theta\left(\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N}\cos 4n\pi r\right)\end{aligned}$$

$$= \begin{cases} \dfrac{1}{\pi} \displaystyle\int_{-\pi/2}^{\pi/2} a_{1/2}^2 \, d\theta = \beta_{1/2} & \text{when } r = \frac{1}{2} \\ 0 & \text{when } \dfrac{1}{\sqrt{6}} \leqslant r < \frac{1}{2} \end{cases} \tag{4.20}$$

The first line of the evaluation follows directly when $\frac{1}{2}$ is substituted for r, and the second line follows when the series is expressed in terms of the real part of sums of powers of complex exponentials. The discontinuity of δ_r at $r = \frac{1}{2}$ can also be interpreted in terms of the results

$$\begin{aligned} &\lim_{r\to\frac{1}{2}} \lim_{N\to\infty} \frac{1}{2N+1}\left[2\,\mathrm{Re}\left\{\frac{\exp[i4(N+1)\pi r]-1}{\exp(i4\pi r)-1}\right\} - 1\right] = 0 \\ &\lim_{N\to\infty} \lim_{r\to\frac{1}{2}} \frac{1}{2N+1}\left[2\,\mathrm{Re}\left\{\frac{\exp[i4(N+1)\pi r]-1}{\exp(i4\pi r)-1}\right\} - 1\right] = 1 \end{aligned} \tag{4.21}$$

When $\delta_{1/2}$ is set equal to $\beta_{1/2}$, the result

$$\mathscr{L} \sim \tfrac{1}{2}\omega^2 \alpha_{1/2} A^2 - \tfrac{1}{2}\beta_{1/2}(A_z{}^2 + A_y{}^2) - \tfrac{1}{4}\gamma_{1/2} A^4 \tag{4.22}$$

is precisely that which is obtained if the plane-wave wave form

$$(\phi, \mathbf{A}) \sim A(\mathbf{x})\, a_{1/2}(\omega t)(0,1,0,0) \tag{4.23}$$

is used to define the one-term approximation. Evidently the discontinuity of δ_r and the corresponding qualitative change in the properties of modulated waves at $r = \frac{1}{2}$ is really there, although no explanation of it has been discerned yet. For present purposes we shall take the safe course, and restrict r to the range $1/\sqrt{6} \leqslant r < \frac{1}{2}$.

Then

$$\mathscr{L} \sim \tfrac{1}{2}\omega^2 \alpha_r A^2 - \tfrac{1}{2}\beta_r(A_z{}^2 + \tfrac{1}{2}(A_x{}^2 + A_y{}^2)) - \tfrac{1}{4}\gamma_r A^4 \tag{4.24}$$

and it may be noted that plane-polarized waves have not been altogether expunged from the theory—we may consider the limit where $r \to \frac{1}{2}$, and there is also the circularly polarized wave seen edgewise in the laboratory frame.

It may be noted again that the coefficients α_r, β_r and γ_r are positive for all values of r in the range $1/\sqrt{6} \leqslant r < \frac{1}{2}$, and furthermore they do not vary much with r. Accordingly, the qualitative features of solutions governed by (4.24) can be discussed in terms of a typical case where r is chosen for convenience. The easiest case is clearly $r = 1/\sqrt{6}$, and for this case it is convenient to change the normalization implied in (4.16) slightly and work with the one-term representation where

$$(\phi, \mathbf{A}) \sim A(\mathbf{x})(0, \cos\theta, \sin\theta, 0) \tag{4.25}$$

and

$$\mathscr{L} \sim \tfrac{1}{2}\omega^2 A^2 - \tfrac{1}{2}(A_z{}^2 + \tfrac{1}{2}(A_x{}^2 + A_y{}^2)) - \tfrac{1}{4}A^4 \tag{4.26}$$

In this paper only two particular solutions of the Euler equations of (4.26) will be discussed—in the first, the solution has spheroidal symmetry and it is localized in space, and in the second, the solution has elliptical symmetry and is localized in the directions normal to the direction of propagation of the wave-frame relative to the laboratory.

The first case is the easier one—in terms of the independent variables $(\bar{x}, \bar{y}, \bar{z}) = (x\sqrt{2}, y\sqrt{2}, z)$, the Euler equation of (4.26) is

$$A_{\bar{z}\bar{z}} + A_{\bar{x}\bar{x}} + A_{\bar{y}\bar{y}} + (\omega^2 - A^2)A = 0 \tag{4.27}$$

and the spherically symmetric version of it (spheroidally symmetric in the original variables) is

$$A_{\bar{r}\bar{r}} + \frac{2}{\bar{r}}A_{\bar{r}} + (\omega^2 - A^2)A = 0 \tag{4.28}$$

where

$$\bar{r} = (\bar{x}^2 + \bar{y}^2 + \bar{z}^2)^{1/2} = (2(x^2 + y^2) + z^2)^{1/2} \tag{4.29}$$

To see that (4.28) has localized solutions we can construct the energy functional

$$\mathscr{E}(\bar{r}) \equiv \tfrac{1}{2}A_{\bar{r}}{}^2 + \tfrac{1}{2}\omega^2 A^2 - \tfrac{1}{4}A^4 \tag{4.30}$$

which, it may be noted, would be the energy integral of (4.28) but for the presence of the term $2A_{\bar{r}}/\bar{r}$. Then it follows directly by the use of the differential equation that

$$\mathscr{E}'(r) = -\frac{2}{\bar{r}}A_{\bar{r}}{}^2 \leqslant 0 \tag{4.31}$$

Once again, the dimensional group of $((AA)^2)_4$ can be used to eliminate a parameter—the substitutions

$$\omega\bar{r} = R, \qquad A = \omega a \qquad \text{and} \qquad \mathscr{E} = \omega^4 \varepsilon \tag{4.32}$$

reduce the problem at hand to the pair of equations

$$\varepsilon(R) = \tfrac{1}{2}a_R{}^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}a^4, \qquad \varepsilon'(R) = -\frac{2}{R}a_R{}^2 \tag{4.33}$$

The qualitative features of the spheroidal wavepackets can now be seen by plotting the lines where ε is a constant in the phase plane (a_R versus a). According to the second of equations (4.33), the trace of a solution crosses those lines in such a manner that the value of ε decreases as R increases, making a first-order contract at points off the a-axis when R is finite and a third order contact when a_R is zero. Figure 8 is a schematic diagram of a typical trace, and the corresponding solution.

Although no specific solution has been drawn in Figure 8, an attempt has been made to exhibit, without undue exaggeration, two features of (4.33) that require further explanation. The first is that solutions that are trapped in the interior of the lens-shaped region defined by $\varepsilon = \frac{1}{4}$ do in fact spiral all the way to the origin of the phase plane as $R \to \infty$. The second feature is

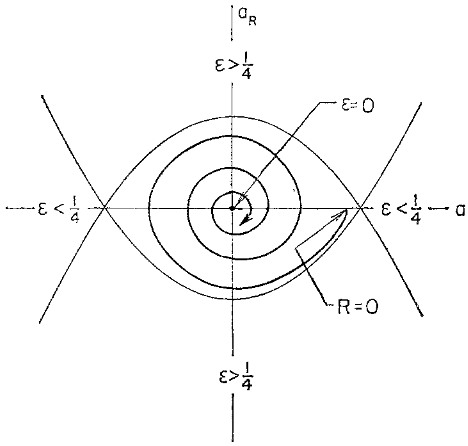


Figure 8a—The trace of a typical solution, $0 < \epsilon(0) < \frac{1}{4}$.

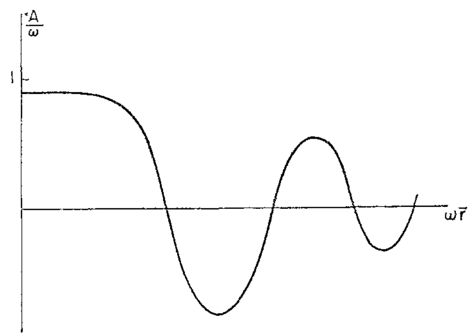


Figure 8b—Qualitative behavior of a typical solution, $A(0) < \omega$.

that solutions that begin near one of the two hyperbolic points where $a = \pm 1$ and $a_R = 0$ tend to hover there until R is relatively large before they spiral to the origin. These and other asymptotic features of the solutions have been discussed by Bisshopp (1971c).

A solution that has spheroidal symmetry in the wave-frame, where it is a purely time-like oscillation, generally has ellipsoidal symmetry in the laboratory frame if the wave-frame is moving relatively to it. The solution can be expressed in the laboratory in terms of a Lorentz transformation, and for present purposes it will suffice to exhibit the case where the wavepacket is moving in the direction of its polarization vector, i.e. where $\Theta = 0$. Then

$$\bar{z} = \frac{z - \beta t}{\sqrt{(1-\beta^2)}}, \qquad \bar{t} = \frac{t - \beta z}{\sqrt{(1-\beta^2)}}, \qquad |\beta| < 1 \tag{4.34}$$

and the solution

$$(\phi, \mathbf{A}) \sim \omega a\left(\omega\left(2(x^2+y^2) + \frac{(z-\beta t)^2}{1-\beta^2}\right)^{1/2}\right)\cdot\left(0, \cos\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}},\right.$$
$$\left.\sin\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}}, 0\right) \tag{4.35}$$

is a wavepacket (still spheroidal in this case) that moves with the group velocity β and is composed of rotating plane waves that travel through it with the phase velocity $1/\beta$. In the general case where $\Theta \neq 0$, the Lorentz transformation induces non-zero fields ϕ and $\hat{z}.\mathbf{A}$ in the laboratory, and the wavepacket appears to be carrying an oscillating charge and the corresponding current. Not much is to be gained by exhibiting such solutions at this point, however, since these wavepackets, if they describe particles, describe free particles, and such features as their net charges and their masses are still hidden, not to emerge until interactions are considered.

The second particular solution of the Euler equations of (4.26) to be discussed here is one in which the solution is independent of the coordinate z in some coordinate system related by the Euler angles to the wave-frame, in which the polarization vector is on the $\bar{z}$-axis. To describe the shape of the resulting beam of radiation we may take $\Psi = \Phi_0 = 0$ without loss of generality, and use the representation where

$$(\phi, \mathbf{A}) \sim A(x,y)(0, \cos\theta, \cos\Theta\sin\theta, \sin\Theta\sin\theta) \tag{4.36}$$

in a frame that moves with the velocity β in the direction of the z-axis of the laboratory frame. Then

$$\bar{\mathscr{L}} \sim \tfrac{1}{2}\omega^2 A^2 - \tfrac{1}{2}\left(\tfrac{1}{2}A_x{}^2 + \frac{1+\sin^2\Theta}{2}A_y{}^2\right) - \tfrac{1}{4}A^4 \tag{4.37}$$

and for the elliptically symmetric solution where

$$A(x,y) = \omega a(R), \qquad R = \omega\left(2x^2 + \frac{2}{1+\sin^2\Theta}y^2\right)^{1/2} \tag{4.38}$$

we obtain the pair of equations

$$\varepsilon(R) = \tfrac{1}{2}a_R{}^2 + \tfrac{1}{2}a^2 - \tfrac{1}{4}a^4, \qquad \varepsilon'(R) = \frac{-1}{R}a_R{}^2 \tag{4.39}$$

Qualitatively, the solutions of (4.39) behave similarly to those of (4.33), but the details of the asymptotic behavior are slightly different in the two cases.

The localization of the solution in space is there, and the laboratory-frame expression

$$(\phi, A) \sim \omega a\left(\omega\left(2x^2 + \frac{2}{1+\sin^2\Theta}y^2\right)^{1/2}\right)\cdot\left(\frac{\beta\sin\Theta}{\sqrt{(1-\beta^2)}}\sin\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}},\right.$$
$$\cos\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}}, \cos\Theta\sin\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}}, \frac{\sin\Theta}{\sqrt{(1-\beta^2)}}$$
$$\left.\times\sin\omega\frac{t-\beta z}{\sqrt{(1-\beta^2)}}\right) \tag{4.40}$$

can be interpreted as a model, in $((AA)^2)_4$, of a fully developed, self-focussed laser beam, propagated without change of shape and surrounded by its diffraction rings. The one-parameter family of solutions defined by $a_R(0) = 0$ and $0 < a(0) < 1$ represents beams of varying sizes (the width of the first dark ring measured along the semiminor axis of symmetry is a convenient measure of the size). For a fixed value of ω the width of the beam varies from zero ($a(0) \to 0$) to infinity ($a(0) \to 1$). Given a member of the family defined by a fixed value of $a(0)$, the width of the beam is proportional to ω^{-1}, and in the limit where $\omega \to 0$ and $((AA)^2)_4$ reduces to Maxwell's equations, the width of the beam increases without bound and there is no contradiction with the nonexistence of self-focussed beams governed by Maxwell's equations.

References

Besicovitch, A. S. (1954). *Almost Periodic Functions.* Dover Publications, Chicago.

Bisshopp, Frederic (1971a). *Nonlinear Field Theories and Structural Stability.* Nonr 562(42), T.R. No. 15, Brown University.

Bisshopp, Frederic (1971b). Chimeras. *Annals of Physics*, **65**, 418.

Bisshopp, Frederic (1971c). $(\phi^4)_4$-wavepackets. *Journal of Differential Equations*, **10**, 568.

Cesari, Lamberto (1964). Functional Analysis and Galerkin's Method. *Michigan Mathematical Journal*, **11**, 385.

Dirac, P. M. (1951). A New Classical Theory of Electrons, Part I. *Proceedings of the Royal Society*, Series A, **209**, 291.

Dirac, P. M. (1952). A New Classical Theory of Electrons, Part II. *Proceedings of the Royal Society*, Series A, **212**, 330.

Dirac, P. M. (1954). A New Classical Theory of Electrons, Part III. *Proceedings of the Royal Society*, Series A, **223**, 438.

Goldstein, Herbert (1953). *Classical Mechanics.* Addison Wesley, Cambridge, Mass.

Nambu, Yoichiro (1968). Quantum Electrodynamics in Nonlinear Gauge. *Progress of Theoretical Physics* (Supplement), Extra No., 190.